Definition. A linear operator F on $\mathcal{H}$ is called <u>adjointable</u> if there exists a linear operator F^* on $\mathcal{H}$ s.t. $\langle F(-), - \rangle = \langle -, F^*(-) \rangle$.

Fact. The adjointable operators on $\mathcal{H}$ denoted by $\mathcal{B}_A(\mathcal{H})$ form a C^*-algebra.

An adjointable linear operator is called (<u>generalized</u>) <u>compact</u> if it is in the norm-closure of the operators of the form $\psi \longmapsto \sum_i \psi_i \langle \psi_i', \psi \rangle$.

Another construction of KK through Hilbert modules.

Definition. $KK_1(B, A)$ is a group of homotopy classes of cycles consisting of

1. a Hilbert A-module $\mathcal{H}$
2. a $*$-homomorphism $\varphi : B \longrightarrow \mathcal{B}_A(\mathcal{H})$
3. an operator $F \in \mathcal{B}_A(\mathcal{H})$

s.t. $[F, \varphi(b)]$, $(1 - F^2)\varphi(b)$, $(F - F^*)\varphi(b)$ are compact.

$\Rightarrow \quad KK_1(\mathbb{C}, A) = K_1(A).$ $\blacksquare$

KK_0 through graded Hilbert modules.

Definition. $KK_0(B, A)$ is a group of homotopy classes of cycles consisting of

1. a Hilbert A-module $\mathcal{H}$ with a $\mathbb{Z}/2\mathbb{Z}$-grading
2. a $*$-homomorphism $\varphi : B \longrightarrow \mathcal{B}_A(\mathcal{H})$ even
3. an operator $F \in \mathcal{B}_A(\mathcal{H})$ odd

s.t.

$$[F, \varphi(b)], \quad (1 - F^2)\varphi(b), \quad (F - F^*)\varphi(b)$$

are compact.

Example. $KK_0(\mathbb{C}, B) \cong K_0(B)$.

In particular $KK_0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$. (H, φ, F) s.t. $\varphi = 1$, $F = \begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix}$ where P is a Fredholm operator, and

$$KK_0(\mathbb{C}, \mathbb{C}) \ni [\mathcal{H}, \varphi, F] \longmapsto \operatorname{ind}(P) \in \mathbb{Z}.$$

Equivariant case (Green-Julg theorem).

For a locally compact group acting continuously on C^*-algebras A and B one defines $KK_*^G(B,A)$ by requiring a $\mathbb{Z}/2\mathbb{Z}$-grading preserving continuous G-action on the Hilbert A-module $\mathcal{H}$, a G-equivariant $\rho : B \to \mathcal{B}_A(\mathcal{H})$, and $(gFg^{-1} - F)\rho(b)$ to be compact for $b \in B$.

Theorem. If G is discrete,

$$KK^G(A,\mathbb{C}) \cong KK(G \ltimes A, \mathbb{C}). \quad \square$$

Theorem. If G is compact,

$$KK_*^G(\mathbb{C}, A) \cong KK_*(\mathbb{C}, G \ltimes A). \quad \square$$

Proof. [Sketch of the main technical point]

G cpct $\Rightarrow$ F can be averaged to obtain F which is G-equivariant because $\int_G gFg^{-1}\,d\chi(g) - F$ is compact. Take

$$\mathcal{H} := \ell^2(\mathbb{N}) \otimes L^2(G) \otimes A$$

Then we have the C^*-Morita equivalence

$$\mathcal{K}(\mathcal{H})^G = \mathcal{K}(\ell^2(\mathbb{N})) \otimes G \ltimes A. \quad \square$$

Another solution to Ex. 36 for $* = 1$.

$B = \mathbb{C} \xrightarrow{\varphi} \mathcal{B}_A(\mathcal{H})$, $\mathcal{H}$ a Hilbert module over A.

Adding a degenerate cycle we may achieve $\mathcal{H} = A \otimes H$. Also, if $T \in \mathcal{K}(\mathcal{H})$ then the homotopy $F + tT$, $t \in [0,1]$ preserves the class of F. Therefore

$$(*) \qquad KK_1(\mathbb{C}, A) = \pi_0\left(\left\{ \dot{F} \in \mathcal{B}_A(A \otimes H)/A \otimes K(H) \;\middle|\; \dot{F}^2 = 1,\ \dot{F} = \dot{F}^* \right\}\right)$$

But $\dot{F} \leftrightarrow \dot{p}$ $\left(\dot{p} = \frac{1}{2}(\dot{F}+1),\ \dot{F} = 2\dot{p} - 1 \right)$

The right hand side of $(*)$ is therefore

$$K_0\left(\mathcal{B}_A(A \otimes H)/A \otimes K(H)\right).$$

the extension

$$A \otimes K(H) \rightarrowtail \mathcal{B}_A(A \otimes H) \twoheadrightarrow \mathcal{B}_A(A \otimes H) / A \otimes K(H)$$

leads to a six-term exact sequence

$$\begin{tikzcd}
K_0(A) \cong K_0(A \otimes K(H)) \arrow[r] & K_0(\mathcal{B}_A(A \otimes H)) \arrow[r] & K_0(\mathcal{B}_A(A \otimes H) / A \otimes K(H)) \arrow[d, "\partial"] \\
K_1(\ldots / \ldots) \arrow[u, "\partial"] & K_1(\ldots) \arrow[l] & K_1(A \otimes K(A)) \arrow[l]
\end{tikzcd}$$

$$K_0(\mathcal{B}_A(A \otimes H)) \cong 0, \qquad 0 \cong K_1(\ldots), \qquad K_1(A \otimes K(A)) \cong K_1(A)$$

$$\Rightarrow K_*(\mathcal{B}_A(A \otimes H) / A \otimes K(H)) \cong K_{*+1}(A).$$

Therefore,

$$KK_1(\mathbb{C}, A) \cong KK_0\left(\mathcal{B}_A(A\otimes H) / A\otimes K(H)\right) \cong KK_1(A). \quad \blacksquare$$

KK-class of an elliptic operator

X cpct manifold, $A = C(X)$,

$$D : \Gamma^\infty(X, E^+) \to \Gamma^\infty(X, E^-)$$

elliptic differential operator of order 1,

$H := H^+ \oplus H^-$, $H^\pm := L^2(X, E^\pm)$, $\mathbb{Z}/2\mathbb{Z}$ graded Hilbert space, $\rho : A \to B(H)$ multiplication operator representation, $f : \mathbb{R} \to [-1, 1]$ continuous s.t. $\lim_{x \to \pm\infty} f(x) = \pm 1$, $f(-x) = -f(x)$ (e.g. $f(x) = \frac{x}{\sqrt{1+x^2}}$)

$$F := f(D).$$

This makes sense since D is essentially self-adjoint.

Then $F^2 - 1 = (\chi^2 - 1)(D)$ is compact because $\chi^2 - 1 \in C_0(\mathbb{R})$ and D has compact <u>resolvent</u> $R(z, D) := (D - z)^{-1}$.

Since D is essentially self-adjoint $F^* = F$.

Since D is of order 1 $[D, \rho(a)]$ is bounded for $a \in C^\infty(X) \Rightarrow [F, \rho(a)]$ is compact.

The triple $(\mathcal{H}, F, \rho)$ gives therefore a cycle for $KK_0(C(X), \mathbb{C})$.

Equivariant KK-theory.

$KK^G_*(B,A)$ is defined by requiring an (even) continuous G-action on $\mathcal{H}$, a G-equivariant φ, and $(g.Fg^{-1}-F)\varphi(B)\subset\mathcal{K}(H)$.

Kasparov product. The most important technical feature of KK-theory.

$$KK^G_i(C,B)\times KK^G_j(B,A)\to K^G_{i+j}(C,A)$$

The construction of an operator F in the definition of a cycle in the image of the product is deep. The product is associative and has other nice properties.

Example. (Easy cases of the Kasparov product)

1) $\varphi : B \to A$ $*$-homomorphism yields a class for $KK_0(B,A)$ by: $\mathcal{H}^+ := A$, $\mathcal{H}^- := 0$, $\varphi : B \to A \lhd \mathcal{B}(A)$, $F := 0$. The Kasparov product with the class of $(\mathcal{H}, F, \varphi)$ in $KK_0(B,A)$ reduces to functoriality of $B \mapsto KK_*(C,B)$, $A \mapsto KK_*(A,C)$ with respect to $*$-homomorphisms $B \to A$.

Let $(\mathcal{H}, F, \varphi)$ be a cycle for $KK_*(B, A)$, $\beta : B' \longrightarrow B$ a $*$-homomorphism, then

$$\beta^*(\mathcal{H}, F, \varphi) = (\mathcal{H}, F, \varphi \circ \alpha)$$

is a cycle of $KK_*(B', A)$.

For $\alpha : A \longrightarrow A'$ one defines

$$\alpha_*(\mathcal{H}, F, \varphi) = (\mathcal{H} \otimes_A A', F \otimes_A A', \varphi \otimes_A A').$$

2) $p = p^* = p^2 \in B$ can be viewed as a $*$-homomorphism $\mathbb{C} \xrightarrow{\pi} B$, $1 \longmapsto p$.

Given a cycle $(\mathcal{H}, F, \varphi)$ for $KK_*(B, A)$ the Kasparov product with $[p] \in K_0(B) = KK_0(\mathbb{C}, B)$ is the cycle $(\mathcal{H}, F, \varphi \circ \pi)$ for $KK_*(\mathbb{C}, A) = K_*(A)$.

Duality in KK-theory.

Idea: If X is a compact manifold, then we expect an adjunction

$$KK_*(B \otimes C(X), A) \cong KK_*(B, C_0(TX) \otimes A),$$

in particular

$$K_*(X) = KK_*(C(X), \mathbb{C}) \cong KK_*(\mathbb{C}, C_0(TX)) \cong K^*(TX)$$

Let β on the right hand side be the image of 1 on the left. Taking $B = C_0(TX)$, $A = \mathbb{C}$ we have

$$KK(C(X) \otimes C_0(TX), \mathbb{C}) \cong KK(C_0(TX), C_0(TX)).$$

Let α on left hand side be the preimage of 1 on the right.

Theorem. Let $\beta \in KK^*(\mathbb{C}, D \otimes \tilde{D})$.

TFAE

(i) The map $KK^G(B \otimes D, A) \to KK^G(B, A \otimes \tilde{D})$ obtained as a composition

$$\begin{tikzcd}
KK^G(B \otimes D, A) \arrow[rr] \arrow[dr, "- \otimes \tilde{D}"'] & & KK^G(B, A \otimes \tilde{D}) \\
& KK^G(B \otimes D \otimes \tilde{D}, A \otimes \tilde{D}) \arrow[ur, "(\mathrm{id}_B \otimes \beta)^*"'] &
\end{tikzcd}$$

is an isomorphism.

(ii) There exists $\alpha \in KK^G(\tilde{D} \otimes D, \mathbb{C})$ such that the following compositions are identities

$$D \xrightarrow{\beta \otimes D} D \otimes \tilde{D} \otimes D \xrightarrow{\alpha \otimes D} D$$

$$\tilde{D} \xrightarrow{\beta \otimes \tilde{D}} \tilde{D} \otimes D \otimes \tilde{D} \xrightarrow{\tilde{D} \otimes \alpha} \tilde{D}$$

Remark. The above means that we have a pair of adjoint functors $(- \otimes D) \dashv (- \otimes \tilde{D})$.

What are α and β for $C(X)$ and $C_0(TX)$?

More generally: how to generate interesting KK-classes between commutative algebras?

Geometric examples.

(i) a proper G-map $f: X \to Y$ induces $f^*: C_0(Y) \to C_0(X)$

(ii) a G-equivariant vector bundle E over X gives a class in $KK^G(C_0(X), C_0(X))$

$\Gamma_0(X, E) \rightsquigarrow$ Hilbert $C_0(X)$-module structure with respect to a fibrewise inner product,

$C_0(X) \longrightarrow \mathcal{K}(\Gamma_0(X,E))$, $F=0$, grading trivial.

(iii) U open in X, $C_0(U) \hookrightarrow C_0(X)$ ideal (extension by zero)

(iv) $V \longrightarrow X$ K-oriented vector bundle over X, the Thom isomorphism gives an invertible element in $KK^G(C_0(V), C_0(X))$.

(K-oriented means that there is a Thom isomorphism in K-theory)